

Dirac equation for quasi-particles in graphene in an external electromagnetic field and chiral anomaly

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There is evidence for existence of massless Dirac quasi-particles in graphene, which satisfy Dirac equation in (1+2) dimensions near the so called Dirac points which lie at the corners at the graphene's brilluoin zone. It is shown that parity operator in (1+2) dimensions play an interesting role and can be used for defining conserved chiral currents [there is no γ^5 in (1+2) dimensions]. It is shown that the "anomalous" current induced by an external gauge field can be related to the anomalous divergence of an axial vector current which arises due to quantum radiative corrections provided by triangular loop Feynman diagrams in analogy with the corresponding axial anomaly in (1+3) dimensions.

Recent progress in the experimental realization of a single layer problem of graphene has lead to extensive exploration of electronic properties in this system. Experimental and theoretical studies have shown that the nature of quasiparticles in these two-dimensional system are very different from those of the conventional two-dimensional electron gas (2DEG) system realized in the semiconductor heterostructures. Graphene has a honeycomb lattice of carbon atoms. The quasiparticles in graphene have a band structure in which electron and hole bands touch at two points in the Brillouin zone. At these Dirac points, the quasiparticles obey the massless Dirac equation in (1+2) dimensions [1]. In other words, they behave as massless Dirac fermions leading to a linear dispersion relation $\epsilon_k = vk$ (with the characteristic velocity $v \simeq 10^6$ m/s). This difference in the nature of the quasiparticles in graphene from conventional 2DEG has given rise to a host of new and unusual phenomena such as anomalous quantum Hall effects and a π Berry phase [1, 2]. These transport experiments have shown results in agreement with the presence of Dirac fermions. The 2D Dirac-like spectrum was confirmed recently by cyclotron resonance measurements and also by angle resolved photoelectron spectroscopy (ARPEC) measurements in monolayer graphene [3]. Recent theoretical work on graphene multilayer has also shown the existence of Dirac electrons with a linear energy spectrum in monolayer graphene [4].

The Dirac points lie at the corners of the graphene's Brillouin zone and have their position vectors in momentum space as [5].

$$\mathbf{K} = \frac{2\pi}{3a}(1, \sqrt{3}), \quad \mathbf{K}' = \frac{2\pi}{3a}(1, -\sqrt{3})$$

where a is the carbon-carbon distance.

Near the Dirac point \mathbf{K}' , the Dirac equation takes the covariant form [6]

$$i(\gamma^\mu \partial_\mu) \psi = 0 \tag{1}$$

where

$$\partial_0 = \frac{1}{v_f} \frac{\partial}{\partial t}$$

and

$$\gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \gamma^1 \partial_1 + \gamma^2 \partial_2 \tag{2}$$

Now it is known [7] that in 3 space-time dimensions there exists two inequivalent representations for γ -matrices [this is true for any odd number of space-time dimensions]:

$$\begin{aligned} \gamma^0 &= \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^2 \\ \gamma^0 &= \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = -i\sigma^2 \end{aligned} \tag{3}$$

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One can take the second representation for the Dirac equation near the Dirac point K, which is obtained from K' by the parity operation:

$$x^1 \longleftrightarrow x^1, x^2 \longleftrightarrow -x^2$$

Taking the two representations mentioned above into account one can write the parity conserving Lagrangian as

$$\mathcal{L} = \bar{\psi}_+(i\partial)\psi_+ + \bar{\psi}_-(i\tilde{\partial})\psi_-$$

where

$$\begin{aligned}\partial &= \gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2\partial_2 \\ \tilde{\partial} &= \gamma^0\partial_0 + \gamma^1\partial_1 - \gamma^2\partial_2\end{aligned}\tag{4}$$

Parity operation takes the solutions in one representation to the other:

$$\begin{aligned}\psi_+^p(x^p) &= -\eta_p\psi_-(x) \\ \psi_-^p(x^p) &= -\eta_p\psi_+(x)\end{aligned}\tag{5}$$

where $x^p = (x^0, x^1, -x^2)$. It is convenient to transform to new fields [7]

$$\begin{aligned}\psi_A &= \psi_+ \\ \psi_B &= i\gamma^2\psi_-\end{aligned}\tag{6}$$

The Lagrangian (4) can then be written as

$$\mathcal{L} = \bar{\psi}_A(i\gamma^\mu\partial_\mu)\psi_A + \bar{\psi}_B(i\gamma^\mu\partial_\mu)\psi_B\tag{7}$$

It is instructive to put mass term in the Lagrangian (7), which one can always put equal to zero:

$$\mathcal{L} = \bar{\psi}_A(i\gamma^\mu\partial_\mu)\psi_A + \bar{\psi}_B(i\gamma^\mu\partial_\mu)\psi_B - mv_f(\bar{\psi}_A\psi_A - \bar{\psi}_B\psi_B)\tag{8}$$

where under the parity operation

$$\psi_{A,B}^P(x^P) = \eta_P\sigma^2\psi_{B,A}(x^P)\tag{9}$$

The Hamiltonian density is

$$\mathcal{H} = v_f[\bar{\psi}_A(-i\gamma^i\partial_i)\psi_A + \bar{\psi}_B(-i\gamma^i\partial_i)\psi_B + mv_f(\bar{\psi}_A\psi_A - \bar{\psi}_B\psi_B)]\tag{10}$$

The Hamiltonian density has the so called conjugate symmetry [8], $\psi_A \leftrightarrow \sigma^3\psi_B$, in the sense that $\mathcal{H} \rightarrow -\mathcal{H}$.

It may be noted that the Lagrangian (8) is invariant, even in the presence of the mass term, two independent transformations

$$\psi_A \rightarrow e^{i\alpha_A}\psi_A, \psi_B \rightarrow e^{i\alpha_B}\psi_B\tag{11}$$

where α_A and α_B are real, and has thus $U_A(1) \otimes U_B(1)$ symmetry. The corresponding conserved currents are

$$J_A^\mu = \bar{\psi}_A\gamma^\mu\psi_A, J_B^\mu = \bar{\psi}_B\gamma^\mu\psi_B\tag{12}$$

One can form even (odd) combination corresponding to "vector" ("axial vector") under parity

$$J_\pm^\mu = 1/2[\bar{\psi}_A\gamma^\mu\psi_A \pm \bar{\psi}_B\gamma^\mu\psi_B]\tag{13}$$

In (1+2) dimensions, there is no γ^5 available as in (1+3) dimensions. But still one may generate "chiral" currents. In fact under the infinitesimal transformations [7].

$$\psi_{A,B} \rightarrow \psi'_{A,B} = \psi_{A,B} + i\alpha\psi_{B,A}$$

and

$$\psi_{A,B} \rightarrow \psi'_{A,B} = \psi_{A,B} \pm \alpha\psi_{B,A}$$

the Lagrangian (8) respectively transforms into

$$\begin{aligned}\mathcal{L} \rightarrow \mathcal{L}_1 &= \mathcal{L} - 2imv_f \alpha (\bar{\psi}_A \psi_B - \bar{\psi}_B \psi_A) \\ \mathcal{L} \rightarrow \mathcal{L}_2 &= \mathcal{L} - 2mv_f (\bar{\psi}_A \psi_B + \bar{\psi}_B \psi_A)\end{aligned}\quad (14)$$

The corresponding conserved "chiral" currents in the absence of mass are

$$J_3^\mu = \frac{1}{2}(\bar{\psi}_A \gamma^\mu \psi_B + \bar{\psi}_B \gamma^\mu \psi_A) \quad (15)$$

$$J_5^\mu = -i\frac{1}{2}(\bar{\psi}_A \gamma^\mu \psi_B - \bar{\psi}_B \gamma^\mu \psi_A) \quad (16)$$

These currents respectively correspond to "vector" and "axial vector" under parity. In the presence of mass term in the Lagrangian (8)

$$\partial_\mu J_3^\mu = imv_f (\bar{\psi}_A \psi_B - \bar{\psi}_B \psi_A) \quad (17)$$

$$\partial_\mu J_5^\mu = mv_f (\bar{\psi}_A \psi_B + \bar{\psi}_B \psi_A) \quad (18)$$

An external gauge field A_μ (electromagnetic) can be introduced by replacing the ordinary derivative by the covariant derivative

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + \frac{ie}{c} A_\mu \quad (19)$$

This gives the Dirac equation in (1+2) dimensions

$$[i\gamma^\mu D_\mu \mp mv_f] \psi_{A, B} = 0, \quad (20)$$

By multiply on the left by $(-i\gamma^\nu D_\nu \mp mv_f)$ one can put the resulting equation in the Pauli form

$$[D^\mu D_\mu + \frac{e}{2c} \sigma^{\mu\nu} F_{\mu\nu} + m^2 v_f^2] \psi_{A, B} = 0 \quad (21)$$

This equation differs from the Klein Gordon equation in the term $\frac{e}{c} \sigma^{\mu\nu} F_{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Using

$$\sigma^{\mu\nu} = \epsilon^{\lambda\mu\nu} \gamma_\lambda \quad (22)$$

one can write

$$\frac{e}{2c} \sigma^{\mu\nu} F_{\mu\nu} = 1/2 f^\lambda \gamma_\lambda \quad (23)$$

where f^λ is current induced by the external gauge field A_μ [8, 9].

$$f^\lambda = \frac{e}{c} \epsilon^{\lambda\mu\nu} F_{\mu\nu} \quad (24)$$

and has abnormal parity. The corresponding induced charge is

$$Q = \int d^3x f^0(x) \quad (25)$$

where

$$f^0(x) = \frac{e}{c} \epsilon^{ij} F_{ij} = 2\frac{e}{c} (\vec{\nabla} \times \vec{A})_3 = 2\frac{e}{c} B \quad (26)$$

Here B is the magnetic field perpendicular to the $x - y$ plane. Thus $Q = \frac{e}{c} \Phi$, where Φ is the magnetic flux.

The parity conserving Lagrangian which gives Eq. (21) is

$$\mathcal{L}_\pm = [\bar{\psi}_A (D^\mu D_\mu + m^2 v_f^2) \psi_A - \bar{\psi}_B (D^\mu D_\mu + m^2 v_f^2) \psi_B] + 1/2 [(\bar{\psi}_A \gamma^\mu \psi_A - \bar{\psi}_B \gamma^\mu \psi_B)] f_\mu$$

so that f_μ is coupled with the current J_-^μ given in Eq. (13) [8]. The Lagrangian is invariant under $U_A(1) \otimes U_B(1)$.

Next we discuss whether f_μ can be related to "anomalous" divergence of some axial current, which arises due to quantum corrections. Obviously such a current can not be J_-^μ . Then the remaining axial current is J_5^μ given in Eq. (16b). Here analogy with the axial vector "anomalous" divergence in (1+3) dimensions, namely [10]

$$\begin{aligned}\partial_\mu J_5^\mu &= -\frac{e^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \\ &= \frac{e^2}{4\pi^2} \vec{E} \cdot \vec{B}\end{aligned}\quad (27)$$

is useful. As is well known this divergence arises from quantum corrections provided by triangle graph which has two vector vertices and one axial vector vertex or that provided by its divergence

$$\partial_\mu J_5^\mu = mc(\bar{u}i\gamma_5 u - \bar{d}i\gamma_5 d)$$

where u and d denote up and down quarks (m is the quark mass), which provide the internal legs of the triangle. Note that although quark mass m appears above, but Eq. (27) is independent of quark mass. In our case we have corresponding ψ_A and ψ_B fields, which appear in the Lagrangian (8) or Hamiltonian (10). Noting that $\partial_\mu J_5^\mu$ involves $m\bar{\psi}_A\psi_B$ and $m\bar{\psi}_B\psi_A$, the relevant Feynman graphs are shown in Fig 1

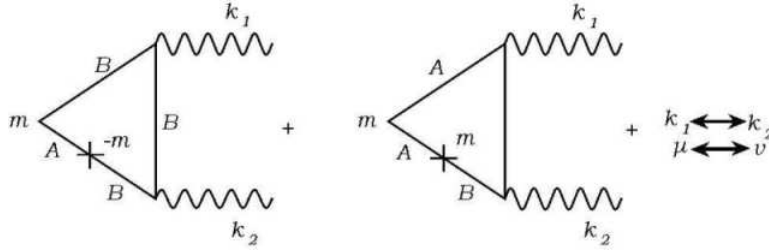


FIG. 1: Triangle diagrams for "anomalous" current divergence.

Note that it is essential to put mass transitions shown (see analogy with Majorana neutrinos), such mass transitions are provided by the divergence of $\partial_\mu J_3^\mu$ given in Eq. (17). Noting that A and B propagators involve opposite masses, the matrix elements are given by

$$\begin{aligned}T_{\mu\nu} &= mv_f e^2 \int \frac{d^D l}{(2\pi)^D} \{Tr[\frac{1}{\not{l} + k_1 - mv_f} \gamma_\mu \frac{1}{\not{l} - mv_f} \gamma_\nu (\frac{1}{\not{l} - k_2 - mv_f} mv_f \frac{1}{\not{l} - k_2 + mv_f}) \\ &\quad + \frac{1}{\not{l} + k_1 + mv_f} \gamma_\mu \frac{1}{\not{l} + mv_f} \gamma_\nu (\frac{1}{\not{l} - k_2 + mv_f} (-mv_f) \frac{1}{\not{l} - k_2 - mv_f})]\} \\ &= m^2 v_f^2 e^2 \int \frac{d^D l}{(2\pi)^D} \frac{Tr[(\not{l} - k_1 + mv_f) \gamma^\mu (\not{l} + mv_f) \gamma^\nu - (\not{l} + k_1 - mv_f) \gamma^\mu (\not{l} - mv_f) \gamma^\nu]}{[(l + k_1)^2 - m^2 v_f^2][l^2 - m^2 v_f^2][(l - k_2)^2 - m^2 v_f^2]} + \frac{k_1 \leftrightarrow k_2}{\mu \leftrightarrow \nu}\end{aligned}\quad (28)$$

The numerator in the integral which contributes is,

$$\begin{aligned}N^\mu &= 2mv_f Tr[\gamma^\rho \gamma^\mu \gamma^\nu (l + k_1)_\rho + \gamma^\rho \gamma^\mu \gamma^\nu l_\rho] \\ &= -4mv_f i[\epsilon^{\rho\mu\nu} (l + k_1)_\rho + \epsilon^{\mu\rho\nu} l_\rho] \\ &\quad - 4mv_f i \epsilon^{\mu\nu\rho} k_{1\rho}\end{aligned}\quad (29)$$

Using the Feynman parametrization, the denominator takes the form,

$$[(l + k_1 x - k_2 y)^2 - \Delta]^3$$

where

$$\Delta = m^2 v_f^2 - x k_1^2 - y k_2^2 + (x k_1 - y k_2)^2$$

Making the shift $l \rightarrow l - (k_1 x - k_2 y)$, the denominator becomes $(l^2 - \Delta)^3$ and the dimensional regularization gives [D=3]

$$\begin{aligned}T_{\mu\nu} &= -4m^3 v_f^3 e^2 i \epsilon^{\mu\nu\rho} k_{1\rho} \int_0^1 dx \int_0^{1-x} dy \frac{(-1)^3 i}{(4\pi)^{d/2}} \frac{\Gamma(3 - D/2)}{\Gamma(3)} \left(\frac{1}{\Delta}\right)^{3-D/2} \\ &\quad + \frac{k_1 \leftrightarrow k_2}{\mu \leftrightarrow \nu}\end{aligned}\quad (30)$$

For photons on the mass shell

$$k_1^2 = 0, \quad k_2^2 = 0$$

and putting $2k_1.k_2 = q^2$, $\Delta = m^2 v_f^2 - q^2 xy$. Thus we obtain, neglecting terms of order q^2 .

$$\begin{aligned} T_{\mu\nu} &= \frac{e^2}{16\pi} \frac{m^3}{(m^6)^{1/2}} \epsilon^{\mu\nu\rho} k_{1\rho} + \frac{k_1 \leftrightarrow k_2}{\mu \leftrightarrow \nu} \\ &= \frac{e^2}{16\pi} (\text{sign } m) \epsilon^{\mu\nu\rho} (k_1 - k_2)_\rho \end{aligned} \quad (31)$$

Thus finally

$$\partial_\mu J_5^\mu = \frac{e^2}{16\pi} (\text{sign } m) \varepsilon_\mu(k_1) \varepsilon_\nu(k_2) \epsilon^{\mu\nu\rho} (k_1 - k_2)_\rho \quad (32)$$

It may be noted while in (1+3) dimensions $\partial_\mu J_5^\mu$ is independent of m ; in (1+2) dimensions, the corresponding quantity $\partial_\mu J_5^\mu$ is also independent of the magnitude of m but does depend on its sign (which is typical for odd space-time dimensions). In both cases mass was used as a regulator. In configuration space Eq. (32) takes the form

$$\partial_\mu J_5^\mu = \frac{e^2}{16\pi} (\text{sign } m) A_\lambda f^\lambda = \frac{e^2}{16\pi} (\text{sign } m) \epsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda \quad (33)$$

[comparing with Eq. (27) in (1+3) dimensions]. Now in terms of electric and magnetic fields.

$$\epsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda = -2A^0 B^3 - 2(\vec{A} \times \vec{E})^3 = -2B[A_0 + \vec{E} \cdot \vec{r}]$$

where B is along z -axis and we have used $\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$ so that $\vec{B} = \vec{\nabla} \times \vec{A}$. One can select a gauge in which $A_0 = 0$ then Eq. (31) becomes

$$\partial_\mu J_5^\mu = -\frac{e^2}{8\pi} (\text{sign } m) B(\vec{E} \cdot \vec{r}) \quad (34)$$

[compare with second line ($\vec{E} \cdot \vec{B}$) of Eq. (27) for axial anomaly in (1+3)]. It is instructive to also calculate $\partial_\mu J_3^\mu$ from the Δ -graph, the only change one has to make is to change the overall sign of the second term in Eq. (28) and m multiplying the integral to im . This changes N^μ in Eq. (29) to

$$\begin{aligned} N^\mu &= 2 \text{Tr}[\gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu (l + k_1)^\rho l^\sigma + m^2 v_f^2 \gamma^\mu \gamma^\nu] \\ &= 4[(l + k_1)^\mu l^\nu + (l + k_1)^\nu l^\mu - (l + k_1)l g^{\mu\nu} + m^2 v_f^2 g^{\mu\nu}] \end{aligned} \quad (35)$$

After making the shift $l \rightarrow l - (k_1 x - k_2 y)$ and noting that terms linear in l do not contribute, one obtains

$$\begin{aligned} T_{\mu\nu} &= 4im^2 v_f^2 \left\{ \frac{i}{(4\pi)^{d/2}} \frac{1}{\Gamma(3)} \int_0^1 dx \int_0^{1-x} dy g^{\mu\nu} \frac{1-d}{2} \Gamma(2-d/2) \left(\frac{1}{\Delta}\right)^{2-d/2} + k_2^\mu k_1^\nu xy \Gamma(3-d/2) \left(\frac{1}{\Delta}\right)^{3-d/2} \right\} \\ &\quad + \frac{k_1 \leftrightarrow k_2}{\mu \leftrightarrow \nu} \end{aligned} \quad (36)$$

where we have put $k_1^2 = 0 = k_2^2$, $k_1.\epsilon_1 = k_2.\epsilon_2 = 0$ and then neglecting $2k_1.k_2 = q^2$, so that $\Delta = m^2 v_f^2$. Then finally

$$\partial_\mu J_3^\mu = \frac{e^2}{4\pi} (\text{sign } m) \{ m v_f \epsilon_1.\epsilon_2 + \frac{1}{4 m v_f} (k_2.\epsilon_1 k_1.\epsilon_2) \} \quad (37)$$

Here the divergence does depend on m in addition to the (sign m). In terms of electric and magnetic fields given above (time independent \vec{A} and uniform magnetic field) and neglecting the second term in Eq. (37).

$$\partial_\mu J_3^\mu = \frac{e^2}{4\pi} (\text{sign } m) m v_f B^2 r^2 \quad (38)$$

One may take $m = \Delta/v_f^2$ where Δ is energy gap in graphene's band structure, referred to as the Dirac gap [11].

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